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A NOTE ON AUGMENTED DESIGNS

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The optimum number of replications of the control treatment(s) in each of the blocks of an augmented randomized complete block design or comparing test treatments with control treatment(s) has been worked out. This replication number maximises the efficiency per observation of the best linear unbiased estimates (BLUE) of the test treatment vs control treatment contrasts. An augmented randomized complete block design with a single control appearing exactly once in each of the blocks has been shown as A- and MV-optimal for making test treatments-control comparisons in a general block design set up. Such a block design is a connected block design with minimum number of observations. The combinations of number of test treatments and number of blocks for which such A- and MV-optimal designs maximise the efficiency per observation have also been obtained.

Key words : Augmented designs, efficiency per observation, designs with minimum number of observations, A-optimality, MV-optimality

In many agricultural situations, experiments are often conducted where the existing practices called control treatments or check varieties are to be compared with new varieties or germplasm collected through exotic or indegenous collections, called test treatments. The experiment material for test treatments is scarce and it is not possible to replicate them in the design. However, enough material is available for replicating control treatments in the design. These kind of experimental situations encountered by Federer (1955) in screening new strains of sugarcane and soil fumigants used in pineapples. Augmented (Hoonuiaku) designs were introduced by Federer (1956) to fill a need arising in screening new strains of sugarcane at Experimental Station of Hawaiian Sugarcane Planters Association on the basis of agronomic characters other than yield. An augmented experimental design is any standard experimental design in standard (control) treatments to which additional (new or test) treatments have been added. The additional treatments require enlargement of either the complete block or the incomplete block in the block design setting and the complete or incomplete rows and (or) columns in the row-column design settings, etc. The augmented design for 1-way elimination of heterogeneity settings is known as augmented block design. The block sizes in these designs may be unequal. Augmented designs eliminating heterogeneity in two directions are called augmented row-column designs. Federer (1956, 1961) gave the analysis, randomization procedure and construction of these designs by adding the new treatments to the blocks of a randomized complete block (RCB) design and balanced lattice designs. Federer (1963) gave procedures and designs useful for screening material inspection and allocation with a bibliography. Federer and Raghavarao (1975) have given a general theory of augmented designs. They obtained the augmented block designs using RCB designs and linked block designs for 1-way elimination of heterogeneity and augmented row-column designs using a Youden square design. Federer, Nair and Raghavarao (1975) gave general methods of construction of augmented row column designs. They also provided formulae for standard errors of estimable treatment contrasts.

A survey of the literature reveals that generally such experiments are conducted using an augmented randomized complete block design. However, the experimenters would often like to know how many times the control treatments be replicated in each of the blocks so as to maximize the efficiency per observation for making test treatments vs control treatment(s) comparisons? To make the idea clearer, consider the following examples:

METHODOLOGY

Example 1 : Consider an experimental situation where 16 tests are to be compared with a single control treatment. The experiment can be laid out in a block design with four blocks, but there is no scarcity of experimental units and the blocks may contain almost 16 experimental units. The four blocks may have unequal number of experimental units. There is enough material available on control treatment that can be replicated but the test treatments can be replicated only once. We studied three possible designs (D1, D2 and D3) in which the replication of the control in each block was one, two and three, respectively. Thus, in the three designs, the control is replicated four, eight and twelve times. The average variance of the BLUE of the tests vs control contrasts and efficiency per observation for the three designs is as given below :

Design	Variance	Efficiency/ observation
D1: w = 16, u = 1, = 4, r = 1, n = 20	$2\sigma^2$	$(1/40)\ \sigma^{-2}$
D2:w = 16, u = 1, b = 4, r = 2, n = 24	$(3/2)\ \sigma^2$	$(1/36)\ \sigma^{-2}$
D3:w = 16, u = 1, b = 4, r = 3, n = 28	$(4/3)\ \sigma^2$	$(3/112)\ \sigma^{-2}$

In the table w denotes the number of test treatments, r the number of times the control is replicated in each of the blocks, b the total number of blocks, n the total number of experimental units and σ^2 the error variance. One can see that in the above example D2 is better than D1 and D3 according to efficiency per observation criterion. It can also be observed that efficiency per observation doesn't increase with the increase in the number of replication of the control in each block (r).

Example 2 : Consider another experimental situation where 36 test treatments are to be compared with two control treatments. The experiment can be laid out in a block design with three blocks. The blocks may have unequal number of experimental units. There is enough material to have replications on controls but the test treatments can be replicated only once. We studied three possible designs (D1, D2 and D3) in which the replication of the control in each block was one, two and three, respectively. Thus in the three designs each of the control treatments is replicated three, six and nine times. The average variance of the BLUE of the test vs control contrasts and efficiency per observation for D1, D2 and D3 is given in the following table.

Design	Variance	Efficiency per observation
D1 : w = 36, u = 2, b = 3, r = 1, n = 42	$(15/9)\ \sigma^2$	$(1/70)\ \sigma^{-2}$
D2 : w = 36, u = 2, b = 3, r = 2, n = 48	$(12/9)\ \sigma^2$	$(1.64)\ \sigma^{-2}$
D3 : w = 36, u = 2, b = 3, r = 3, n = 54	$(11/9)\ \sigma^2$	$(1.66)\ \sigma^{-2}$

One can see that in the above example D2 is better than D1 and D3 according to efficiency per observation criterion. It can also be observed that efficiency per observation doesn't increase with the increase in the number of replication of the control in each block (r). Therefore, it is important to answer the above question and in section 2, we have made an attempt to answer this question.

RESULTS AND DISCUSSION

Consider an experimental situation in which $w = \sum_{j=1}^{b} k_j$ test treatments are to be compared with u control treatments each replicated r times in each of the b blocks such that j^{th} block size is $k_j + ur$, $j = 1, \ldots, b$. The total number of experimental units are $w + ubr$ and the total number of treatments is $w + u$. If we denote $1, \ldots, w$ as test treatments and $w + 1, \ldots, w + u$ as control treatments, then the coefficient matrix of reduced normal equations for estimating treatment effects is given by :

$$C = \begin{bmatrix} A_1 & 0 & \cdots & 0 & -\frac{r}{k_1 + ur} 1_{k_1} 1_u' \\ 0 & A_2 & \cdots & 0 & -\frac{r}{k_1 + ur} 1_{k_1} 1_u' \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & A_b & -\frac{r}{k_b + ur} 1_{k_b} 1_u' \\ -\frac{r}{k_1 + ur} 1_u 1_{k_1}' & -\frac{r}{k_2 + ur} 1_u 1_{k_2}' & \cdots & -\frac{1}{k_b + ur} 1_u 1_{k_2}' & br\, 1_u - \left(r^2 \sum_{j+1}^{b} \frac{1}{k_j + ur} \right) 1_u 1_u' \end{bmatrix}$$

$$C = \begin{bmatrix} 1_{k_1} + \frac{1}{ur} 1_{k_1} 1_{k_1}' & 0 & \cdots & 0 & 0 \\ 0 & 1_{k_2} + \frac{1}{ur} 1_{k_2} 1_{k_2}' & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1_{k_b} + \frac{1}{ur} 1_{k_b} 1_{k_b}' & 0 \\ 0 & 0 & \cdots & 0 & \frac{1}{br}\left[1_u - \frac{1}{u} 1_u 1_u' \right] \end{bmatrix}$$

The contrasts of main interest are those between test treatments and control treatments and it is easy to see that the design in variance balanced with respect to these contrasts and variance of BLUE of each of these contrasts is given by

$$\text{Var}\,(\hat{\tau}_t - \hat{\tau}_c) = \left(\frac{ubr + b + u - 1}{ubr} \right) \sigma^2;$$

$$\forall t = 1, \ldots, w,\ c = w + 1, \ldots, w + u \quad (2.1)$$

where σ^2 is the error variance.

Therefore, Average variance, $\bar{V} = \sigma^2 \left(\frac{ubr + b + u - 1}{ubr} \right)$

$$\text{Efficiency} = \frac{\sigma^{-2}\, ubr}{ubr + b + u - 1}$$

$$\text{Efficiency/Observation} = \frac{\sigma^{-2}\, ubr}{(ubr + b + u - 1)\,(w + ubr)}$$

$$= \sigma^{-2}\, \varphi\,(b, w, u, r) \qquad \ldots (2.2)$$

For fixed b, u and w, φ behave as a function of r and is independent of $k_1, \ldots, k_b$ separately. It depends on the k_j's only through $w = \sum_{j=1}^{b} k_j$

$$\varphi(b,w,u,r) = \frac{-b - u + 1}{(w - b - u + 1)\,(ubr + b + u - 1)} + \frac{-w}{(-w + b + u - 1)\,(w + ubr)}$$

$$= \frac{1}{w - b - u + 1}\left[\frac{-b - u + 1}{ubr + b + u - 1} + \frac{w}{w + ubr}\right]$$

$$\varphi'(r) = 0$$

$$\Rightarrow \frac{-w}{(w + ubr)^2} + \frac{u + b - 1}{(ubr + b + u - 1)^2} = 0$$

$$\Rightarrow \frac{\sqrt{w}}{w + ubr} = \frac{\sqrt{u + b - 1}}{ubr + b + u - 1}$$

$$\Rightarrow \sqrt{w}\, ubr + \sqrt{w}\,(b + u - 1) = w\sqrt{u + b - 1} + ubr\sqrt{u + b - 1}$$

$$\Rightarrow ubr\,(\sqrt{w} - \sqrt{u + b - 1}) = \sqrt{u + b - 1}\,\sqrt{w}\,(\sqrt{w} - \sqrt{u + b - 1})$$

$$\Rightarrow ubr = \sqrt{u + b - 1}\,\sqrt{w}$$

$$\Rightarrow r = \frac{\sqrt{u + b - 1}\,\sqrt{w}}{ub}$$

$$\Rightarrow r = \frac{1}{u}\sqrt{\frac{u + b - 1}{b}}\,\sqrt{\frac{w}{b}} \qquad \ldots (2.3)$$

$$\varphi''(r)\Big|_{r=\frac{1}{u}\frac{\sqrt{u+b-1}}{b}\sqrt{\frac{w}{b}}} < 0$$

requires that $b+u-1 \leq w$. Therefore, the efficienty per observation is maximized when r is as given in (2.3) provided b + u − 1 $\leq w$. For example, consider the problem of obtaining the optimum number of replications of the controls in an experiment with w = 24, u = 3, b = 4. We have

$$r + \frac{1}{3}\sqrt{\frac{6\times 24}{4\times 4}} = 1.$$

Similarly, for w = 98, u = 2, b = 7, we have

$$r + \frac{1}{3}\sqrt{\frac{8\times 98}{7\times 7}} = 2$$

Remark 2.1 : For a single control situation, i.e. u = 1, the expression (2.3) reduces to $r=\sqrt{\frac{w}{b}}$ and it can easily be seen that for $u=1, b$, which is always true.

There may, however, arise many combinations of w, u and b for which expression (2.3) does not yield a positive integer value of r. In such situations, a question arises as to what integer value of r should be taken. For this efficiency per observation has been calculated of $w \leq 100, b \leq 25$ and $u \leq 10$ such that $b+u-1 \leq w$ and r has been taken as r^* $int(r)$ and $int(r)+1$ in the expression (2.3) besides taking r = 1. A close scrutiny reveals that if value of r > #. 42 then take r^* $int(r)+1$ and for values of r smaller than or equal to #. 42 take r^* $int(r)$ for $u \geq 2$. For u = 1, the same rule applies but the value of r is taken as #. 45 instead of #. 42.

The augmented design with u = 1 and r = 1 are the designs with minimum number of observations i.e., number of observations (n) = number of treatments (w + 1) + number of blocks (b) − 1. These designs can be obtained by allocating control treatment to each of the blocks once and allocating the treatments 1, ..., w to the remaining units of the blocks such that a treatment appears in only one block. For example, a design with w + 1 = 9, b = 4, k = 3 is

9	9	9	9
1	3	5	6
2	4	6	8

These designs have been shown to be A- and MV- optimal for making test treatments-control comparisons in a proper block design set up by Dey, Shah and Das (1995). Through simple algebra on the lines of Lemma 4.1 and Theorem 5.2 of Dey, Shah and Das (1995), it can be shown that the above procedure gives A- and MV-optimal designs for making test treatments control comparisons under a non-proper block design set up as well. At this stage, one may pose a question whether A- and MV-optimal designs for making test treatments-control comparisons also maximizes the efficiency per observation or not. To answer this question, the combinations of w and b which maximizes the efficiency per observation have been obtained through empirical investigations as described earlier for obtaining the value of r, and are given in the following table.

b	w
2	3 to 4
3	4 to 6
4	5 to 8
5	6 to 10
6	7 to 12
7	8 to 14
8	9 to 16
9	10 to 18
10	11 to 20

11	12 to 22
12	13 to 24
13	14 to 26
14	15 to 28
15	16 to 30
16	17 to 32
17	18 to 34
18	19 to 36
19	20 to 38
20	21 to 40
21	22 to 42
22	23 to 44
23	24 to 46
24	25 to 48
25	26 to 50

It is to be noted that analysis of the designs so generated does not pose any problem as a g-inverse of C, C^{-} is given and adjusted treatment total vector can be obtained as per procedure of general block designs and other sum of squares can be obtained as per established procedures and various types of contrasts can be estimated using contrast analysis.

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